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THE RAMSEY NUMBERS $r(P_m, K_n)$

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Abstract. This note evaluates the Ramsey numbers $r(P_m, K_n)$, and discusses developments in generalized Ramsey theory for graphs.

1. Introduction

Let A and B be graphs. Define $r(A, B)$ to be the least integer N such that, for every graph G with N vertices, either G contains A as a subgraph or its complement $\bar{G}$ contains B. The existence of $r(A, B)$ follows from a theorem of Ramsey [20].

Let K_n, P_n, C_n and $K_{1,n}$ denote respectively a complete graph on n vertices, a path with n vertices, a cycle with n vertices and a star of degree n. We follow the notations and definitions of [18].

Until recently, attention was devoted only to the cases $r(K_m, K_n)$, for which few nontrivial numbers are known [17]. The refinement $r(A, B)$, now known as "generalized Ramsey theory for graphs" [7], has already proved fruitful. It was apparently first suggested in a paper of Gerencsér and Gyárfás [16] which established that

$$(1) \qquad r(P_m, P_n) = n + [\tfrac{1}{2}m] - 1 \qquad \text{for } m \leq n .$$

(This result was later rediscovered by Burr and Roberts [2], and by the author.) However, the theory was perhaps implicit in earlier work of Erdös [13], who essentially evaluated $r(P_m, K_n)$ in 1947 by proving Theorem 2 below.

Generalized Ramsey theory was also formulated independently by Cockayne [11], and Chvátal and Harary [7].

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Chartrand and Schuster [3–5, 22] have investigated the numbers $r(C_m, C_n)$, as have Bondy and Erdös [1]. Burr and Roberts [2] evaluated all $r(K_{1,m}, K_{1,n})$, and Schwenk has determined all the numbers $r(K_m, K_{1,n})$ (see [19]). Williamson [23] has computed the Ramsey numbers for paths in directed graphs.

Chvátal and Harary [7–10] have discussed further refinements, and have computed many numbers $r(A, B)$ for both A, B having few vertices. Harary [19] has surveyed recent results.

Some other outgrowths of "Ramseyian" theory appear in [6, 14, 21].

2. The numbers $r(P_m, K_n)$

Theorem 1. $r(P_m, K_n) = (m-1)(n-1) + 1$.

The graph composed of $n-1$ disjoint copies of K_{m-1} contains no P_m and its complement contains no K_n, so $r(P_m, K_n) > (m-1)(n-1)$. Theorem 1 thus follows at once from a theorem of Erdös [13].

Theorem 2 (Erdös). *If a graph G has $(m-1)(n-1) + 1$ vertices, then G contains P_m or $\bar{G}$ contains K_n.*

This theorem is itself an immediate consequence of a theorem of Gallai [15].

Theorem 3 (Gallai [15]). *A graph G is not $(m-1)$-colorable if and only if every orientation of G contains a (directed) path with m vertices.*

Indeed, if G has $(m-1)(n-1) + 1$ vertices and its complement $\bar{G}$ contains no K_n, then G is not $(m-1)$-colorable, so G contains a P_m by Gallai's Theorem.

It is worth mentioning that Theorem 2 also follows quickly from a well-known theorem of Dilworth [12]: Let the vertices of G be $v_1, ..., v_k$ and suppose that G contains no P_m and its complement contains no K_n. Partially order the vertices by letting $v_i \leq v_j$ if either $i = j$ or $i < j$ and there exists a path $v_i, v_{i_1}, ..., v_{i_s}$ in G such that $i < i_1 < ... < i_s = j$. Clearly, all chains have at most $m-1$ points and all antichains have at most $n-1$ points in this ordering, so $k \leq (m-1)(n-1)$ by Dilworth's Theorem, which implies Theorem 2.

3. Other Ramsey numbers involving paths

It is not surprising that $r(P_m, K_n)$ and $r(P_m, P_n)$ were apparently the first nontrivial generalized Ramsey numbers to be determined completely. Numbers of the form $r(P_m, B)$ might be expected to be relatively tractable. The author has recently proved the following theorem.

Theorem 4.

$$r(P_m, K_{1,n}) = \begin{cases} m+n-1 & \text{if } n \equiv 1 \pmod{m-1}, \\ m+n-2 & \text{if } n \equiv 0,2 \pmod{m-1}, \\ m+n-2 & \text{if } n \not\equiv 1 \pmod{m-1} \text{ and } n \geq m^2-6m+9\,, \end{cases}$$

provided that $m \geq 3$. When $m = 2$, the corresponding numbers are trivial.

Regarding the numbers $r(C_m, P_n)$, current work shows that

(2) $\qquad r(C_m, P_n) = 2n-1 \quad \text{for odd } m \leq n.$

Indeed, P. Erdös has written the author that V. Rosta has proved $r(C_m, C_n) \leq 2n-1$ for all $m \leq n$ except $(m, n) = (3,3)$. It follows that for odd $m \leq n$,

$$2n-1 \geq r(C_m, C_n) \geq r(C_m, P_n) > 2n-2\,,$$

where the last inequality comes from the complete bipartite graph $K_{n-1,n-1}$. Additionally, $r(C_3, P_3) = 5$ follows from Theorem 1, and $r(C_3, C_3) = 6$ is well known.

The author has proved $r(C_4, P_n) = n+1$, $n \geq 3$, and $r(C_6, P_n) = n+2$, $n \geq 6$. Except for a few special cases, this follows from the formulas for $r(C_4, C_n)$ and $r(C_6, C_n)$ in [3, 22]. Perhaps it is true that $r(C_m, P_n) = n + [\frac{1}{2}m] - 1$ for even $m \leq n$.

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